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A study on modified Szabo's wave equation modeling of frequency-dependent dissipation in ultrasonic medical imaging

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Abstract
The modified Szabo's wave equation, in which the positive fractional derivative has first been used, is a new model to describe the frequency-dependent dissipative ultrasonic wave propagation through human tissues. To our best knowledge, the verification of this model, however, has not been reported in literature. Based on the frequency-dependent dissipation characterization of tumor and surrounding normal tissues, clinical amplitude velocity reconstruction imaging (CARI) is a recent ultrasonography for effectively detecting early breast tumors. This study makes the first attempt to numerically test the modified Szabo's model to the CARI clinical technique. The finite difference method is employed to solve the modified Szabo's wave equation. It is observed from our experimental results that the reflecting line of ultrasound pressure of the model is enhanced in the region of tumor against surrounding normal tissues. This finding agrees well with clinical observations and shows that the model can well describe the ultrasonic frequency-dependent dissipation. We also note that the numerical solution of positive fractional derivative modified Szabo's wave equation is as expensive as that of the standard fractional derivative equations.

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1. Introduction

For medical imaging of soft human tissues, ultrasonography outperforms the other techniques regarding resolution, contrast mechanism, acceptability, less expense, real-time and non-invasive (non-radiative) features (Wells 2000) and has thus become popular in detecting early tumors, notably breast tumors. In recent years, more than one-fourth of all medical imaging researches are on using ultrasound, and the proportion is ever increasing (WFUMB 1997). For research and development of ultrasonic imaging techniques, nowadays numerical simulation plays a far more important role and is also more cost-effective than the traditional trial–error approach.

Human tissues are often considered as viscoelastic multiphase media. The attenuation of acoustic waves propagating in a wide variety of human tissues is observed to obey the following empirical frequency-dependent dissipative power law (Chen and Holm 2003, Szabo 1994, Szabo and Wu 2000)

$$P(x+\Delta x) = P(x)\mathrm{e}^{-\alpha(\omega)\Delta x}, \quad \alpha = \alpha_0 \left|\omega\right|^{\eta}, \qquad (1)$$

where ω is the angular frequency, P is the pressure, α_0 and η are arbitrary real non-negative constants, x is the displacement and Δx is the wave propagation distance. The accurate mathematical model of such frequency-dependent dissipative acoustic propagation is very important in the quality improvement of medical ultrasonography.

The standard integer-order differential equation model of wave equation cannot reflect such frequency-dependent dissipation. The traditional multi-relaxation model demands some obscure parameters, unavailable from experimental measurements. The time fractional derivative has been used to successfully model such a complex physical process with easily available parameters. But it is noted that the time fractional derivative does not guarantee positivity. Instead Szabo (1994) proposed a causal time domain wave equation model to effectively describe this acoustic frequency-dependent dissipation. However, due to the hypersingular improper integral in Szabo's model, its numerical solution is almost infeasible. Chen and Holm (2003) observed the similarity and nuances between the fractional derivative model (Adolfsson *et al* 2005, Gorenflo *et al* 2002, Magin 2008, Xu and Tan 2006) and the Szabo model (Szabo 1994) of dissipative acoustic wave propagation. Then, based on the concept of the Caputo fractional derivative concept, Chen and Holm (2003) introduced the positive fractional derivative to remedy the hypersingularity drawback and then further proposed the modified Szabo's wave equation.

However, the numerical verification of the modified Szabo's model has not been reported in the literature. This study makes the first attempt to test this model with a recent ultrasonography technique, called clinical amplitude velocity reconstruction imaging (CARI) technique, which was developed to effectively detect early breast tumor (Richter 1995, Richter and Heywang-Köbrunner 1995). Breast cancer is one of the leading causes that lead to death from cancer among American women (American Cancer Society 2008). This study focuses on simulating the CARI imaging of breast tumors via the modified Szabo's wave equation. The finite difference method (FDM) is used to solve the modified Szabo's equation.

The organization of the paper is as follows. In section 2, we introduce the CARI imaging of breast tumors. Section 3 presents the modified Szabo's model and its FDM discretization scheme. In section 4, two-dimensional numerical results are displayed and discussed. We conclude this study with some remarks in section 5.

2. Introduce to CARI technique

In the configuration of CARI, as shown in figure 1, the breast is fixed between two plates as in the mammography-identical position (Richter 1995). The transducer is placed above the upper plate; and the lower plate, a metallic reflecting structure, indicates the result of diagnosis by producing a reflecting line of ultrasound pressure. Tumors can be identified from the benign tissue in terms of changes of attenuation and velocity of the ultrasound in different media (Kossoff *et al* 1973, Richter and Heywang-Köbrunner 1995). Tumors tend to be denser and harder than the benign tissues. Thus, the acoustic wave travels faster and dissipates more energy in tumors. If all of the region of tissue is homogeneous without any lesions, the reflecting line should be straight. On the other hand, if some pathological changes occur, the reflecting line will be heightened in the region of tumor against surrounding normal tissues.

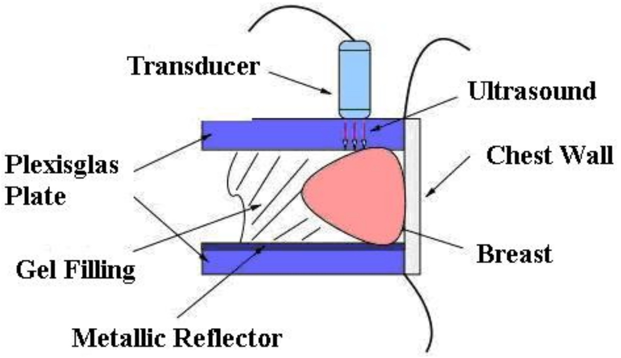

Figure 1. Configuration of the CARI imaging of breast tumor (Bounaïm *et al* 2004a).

3. Mathematical models and numerical discretization schemes

3.1. Modified Szabo's model

By using Fourier transform and the smallness approximation, Szabo (1994) derived a general frequency domain wave equation to describe frequency-dependent acoustic attenuation of arbitrary order η. After considering the causal relations, he proposed the following time domain wave equation:

$$\frac{1}{c_0^2}\frac{\partial^2 P}{\partial t^2}+\frac{2\alpha_0}{c_0}S_y(P)=\Delta P, \tag{2}$$

where

$$S_y(P)=\begin{cases}\partial P/\partial t, & \eta=0,\\ -\dfrac{2\Gamma(\eta+2)\cos[(\eta+1)\pi/2]}{\pi}\displaystyle\int_0^t\frac{P(\tau)}{(t-\tau)^{\eta+2}}\mathrm{d}\tau, & 0<\eta<2,\\ -\partial^3 P/\partial t^3, & \eta=2,\end{cases}$$

Δ denotes the Laplacian, η is a real non-negative constant, c_0 represents the wave speed and $\Gamma(\eta)$ is the Gamma function.

It is noted that the above Szabo's wave equation consists of a hypersingular improper integral. To solve this problem, Chen and Holm (2003) used the fractional derivative concept to introduce the positive fractional derivative and thus derived the following modified Szabo's wave equation:

$$\frac{1}{c_0^2}\frac{\partial^2 P}{\partial t^2}+\frac{2\alpha_0}{c_0}Q_y(P)=\Delta P, \tag{3}$$

where

$$Q_y(P)=\begin{cases}\partial P/\partial t, & \eta=0,\\ \dfrac{\partial^{|\eta|+1}P}{\partial t^{|\eta|+1}}, & 0<\eta<2,\\ -\partial^3 P/\partial t^3, & \eta=2,\end{cases}$$

is used to describe the damping effect, η is the order of the positive fractional derivative, and the positive fractional derivative (Chen and Holm 2003) is defined as

$$\frac{\mathrm{d}^{|\eta|}P}{\mathrm{d}t^{|\eta|}}=\frac{1}{q(\eta)}\int_0^t\frac{P}{(t-\tau)^{\eta+1}}\mathrm{d}\tau, \tag{4}$$

where the constant q is given by

$$q(\eta)=\frac{\pi}{2\Gamma(\eta+1)\cos[(\eta+1)\pi/2]}.$$

The Fourier transform of the positive fractional derivative is positive ($|\omega|^{\eta}P$). Thus, positive fractional derivative can be used to describe well the acoustic attenuation obeying the frequency-dependent power law formula (1) (Chen and Holm 2003).

3.2. Numerical discretization schemes

This subsection gives the FDM discretization formulation of the modified Szabo's wave equation. Based on the following relation:

$$\frac{1}{\Gamma(-\eta)}\int_0^t \frac{P(\tau)}{(t-\tau)^{\eta+1}}\mathrm{d}\tau=\lim_{\Delta t\to 0}\Delta t^{-\eta}\sum_{r=0}^{n}\omega_r^{(\eta)}P(t-r\Delta t), \quad (5)$$

the damping item in equation (3) can be approximated by the finite difference (Podlubny 1999)

$$\begin{aligned}\partial^{|\eta|+1}P/\partial t^{|\eta|+1} &\approx A\Delta t^{-\eta}\sum_{r=0}^{n}\omega_r^{(\eta)}\frac{\partial P_{n-r}}{\partial t}\\ &\approx A\Delta t^{-\eta-1}\sum_{r=0}^{n-1}\omega_r^{(\eta)}(P_{n-r}-P_{n-r-1})+A\Delta t^{-\eta}\omega_n^{(\eta)}P'(0), \quad (6)\end{aligned}$$

where $A=2\Gamma(-\eta)\Gamma(\eta+1)\cos[(\eta+1)\pi/2]/\pi$ and $\omega_r^{(\eta)}=(-1)^r\eta(\eta-1)\mathrm{L}\,(\eta-r+1)/r!$. Equation (3) can be discretized when $0<\eta<2$ and $\eta\neq 1$

$$\begin{aligned}&\frac{1}{c_0^2}\left(\frac{P_{i,j}^{n+1}-2P_{i,j}^{n}+P_{i,j}^{n-1}}{\Delta t^2}\right)+\frac{2\alpha_0 A}{c_0\Delta t^{\eta}}\\ &\times\left[\Delta t^{-1}\sum_{r=0}^{n}\omega_r^{(\eta)}(P_{i,j}^{n-r+1}-P_{i,j}^{n-r})+\omega_{n+1}^{(\eta)}P'(0)_{i,j}\right]\\ &=\frac{P_{i+1,j}^{n}-2P_{i,j}^{n}+P_{i-1,j}^{n}}{\Delta x^2}+\frac{P_{i,j+1}^{n}-2P_{i,j}^{n}+P_{i-1,j+1}^{n}}{\Delta y^2}, \quad (7)\end{aligned}$$

3.3. Initial and boundary conditions

Equation (3) is subjected to certain initial and boundary conditions according to the two-dimensional configuration of figure 2, given in Bounaïm *et al* (2004a, 2004b).

Enclosed by the atmosphere, the tissue is initialized with the following atmosphere pressure condition:

$$P(\mathbf{x},0)=P_{\mathrm{atm}},\qquad \partial P(\mathbf{x},0)/\partial t=0. \quad (8)$$

The transducer is characterized as the Dirichlet condition

$$P(\mathbf{x}_{\mathrm{tran}},t)=P_{\mathrm{tran}}(\mathbf{x},t). \quad (9)$$

The reflecting plate is represented by the reflecting boundary condition

$$\partial P(\mathbf{x},t)/\partial n=0. \quad (10)$$

The first-order absorbing conditions are set on all the other boundaries

$$\partial P(\mathbf{x},t)/\partial n=-[\partial P(\mathbf{x},t)/\partial t]/c_0. \quad (11)$$

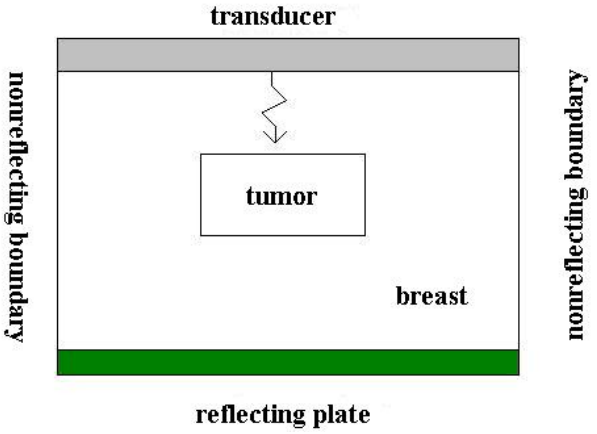

Figure 2. The two-dimensional configuration of the CARI technique of breast tumors (Bounaïm *et al*2004a, 2004b).

Meanwhile, to get stable numerical results, the time step should satisfy a stricter stability condition than the following condition (Bounaïm *et al*2004b):

$$\Delta t\leqslant\frac{1}{c_0}\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)^{-1/2}. \quad (12)$$

4. Numerical results and discussions

To our knowledge, the numerical simulation of the modified Szabo's wave equation has not been reported in the literature. This section focuses on the numerical testing of the modified Szabo's model of the two-dimensional CARI imaging of breast tumors.

The fractional derivative equation is known to be computationally expensive and the positive fractional derivative Szabo equation is no exception. To reduce the computing cost down to an affordable level, we consider small computational domains, namely, 5 mm × 10 mm normal fatty breast tissue embedding a 1 mm × 2 mm tumor in its center. The tested ultrasound is 3.75 MHz. The mean wave speeds and attenuation parameters in the fatty tissue and the tumor are, respectively, $c_{0\mathrm{F}}=1475\,\mathrm{m\,s^{-1}}$, $\alpha_{0\mathrm{F}}=15.8/(2\pi)^{1.7}\,\mathrm{dB\,m^{-1}\,MHz^{-1.7}}$, $\eta_{\mathrm{F}}=1.7$ and $c_{0\mathrm{T}}=1527\,\mathrm{m\,s^{-1}}$, $\alpha_{0\mathrm{T}}=57.0/(2\pi)^{1.3}\,\mathrm{dB\,m^{-1}\,MHz^{-1.3}}$, $\eta_{\mathrm{T}}=1.3$, which were obtained from experimental measurements reported by D'astrous and Foster (1986) and Weiwad *et al* (2000). The wavelength is $\lambda=2c/\omega\approx 0.4$ mm. Since the reflecting plate is used as an indicator for the presence of tumors, we need to compute the traveling time of wave propagation before reflection. In our cases, the time interval is about 3.3 μs.

In terms of the Shannon sampling principle, we should keep the time step no larger than $1/(2f)$, and the grid spacing is about $\frac{1}{2}$–$\frac{1}{10}$ of the wavelength λ to obtain accurate numerical results (Bounaïm *et al*2004a, 2004b). The stable condition as mentioned earlier should also be considered. Consequently, the time step of this simulation is $\Delta t=2.6657\times 10^{-2}\,\mu$s, the spatial grid spacings are $\Delta x=0.05$ mm and $\Delta y=0.1$ mm. Here x denotes the direction of wave propagation and y the reflecting plate. The initial input of the transducer is given as

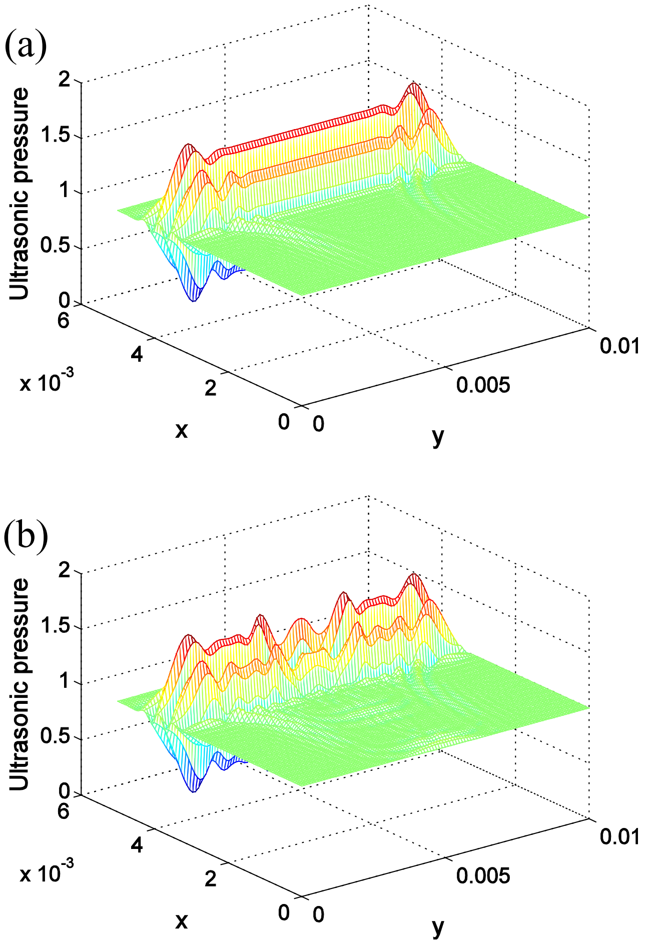

Figure 3. Normalized ultrasound pressure when $t = 3.3\,\mu$s: (a) pressure in the normal breast tissue and (b) pressure in the breast tissue with a tumor.

follows (Bounaïm *et al*2004b):

$$P_{\text{tran}}(\mathbf{x},t) = \begin{cases} \left[\begin{array}{l} P_{\text{atm}} + P_{\text{atm}}\dfrac{\cos[\omega_1(t-0.6\,\mu\text{s})]}{2} \\ \quad \times\{1+\cos[\omega_2(t-0.6\,\mu\text{s})]\}, \end{array}\right] & 0\leqslant t\leqslant 1.2\,\mu\text{s}, \\ P_{\text{atm}}, & t\geqslant 1.2\,\mu\text{s}, \end{cases} \tag{13}$$

where P_{atm} is the standard atmospheric pressure, $\omega_1 = 2\pi f$ and $\omega_2 = \omega_1/4$.

Wave speeds and attenuation coefficients in the normal tissue and the tumor are different and denoted, respectively, by $c_{0\text{F}}$, $\alpha_{0\text{F}}$, η_{F} and $c_{0\text{T}}$, $\alpha_{0\text{T}}$, η_{T}. Numerical results are shown in figures 3 and 4.

From the numerical results shown in figure 3, the profile of normalized ultrasound pressure displays fluctuation when ultrasound travels through the region of tumor, which differs greatly from that of the normal tissue. Additionally, the reflecting line of ultrasound pressure of the normal tissue is straight. Otherwise, if there is a tumor in the breast tissue, as shown in figure 4, the reflecting line is enhanced, which is a recognizable signature of tumor existence inside the fatty tissue. This finding coincides with clinical observations. Our above numerical results suggest that the modified Szabo's wave equation can well describe the frequency-dependent power law of acoustic dissipation in human tissues.

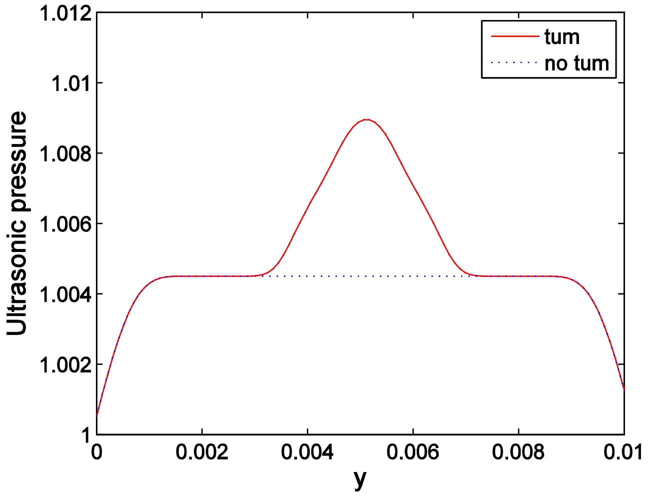

Figure 4. Normalized ultrasound pressure along the reflecting plate ($x = 5$ mm) when $t = 3.3\,\mu$s. The dashed line denotes the reflecting line of the normal breast tissue, whereas the solid line denotes the reflecting line of the breast tissue including a tumor.

5. Conclusions

This paper makes the first attempt to numerically simulate the two-dimensional modified Szabo's wave equation model of CARI breast imaging.

Numerical results show that the enhanced reflecting line of ultrasound wave indicates the suspicious existence of tumor, which agrees well with clinical observations. The simulation verifies that the modified Szabo's model is appropriate and effective to describe the acoustic frequency-dependent dissipation phenomena occurring in the CARI medical imaging.

This study also gives rise to a few research issues worthy of further investigation, among which are

- The geometry of the breast tissue is truncated into rectangles and the tumor is also assumed rectangular. But the tumor is known to have irregular domain and is not necessarily rectangular. For a complex-shaped domain, the FDM may not be the most appropriate numerical technique.
- This study only investigated the two-dimensional modified Szabo's wave equation. Numerical simulation of three-dimensional cases is still under way and will be reported in a subsequent paper.
- The physics mechanism underlying the phenomenological modified Szabo's model of acoustic dissipation in complex soft media is largely unclear. In order to find a better characterization, our further work will focus on comparing the modified Szabo's model with other models, such as fractal, fractional and fractional Laplace models.
- The fractional derivative, in fact, is a convolution operator and the computing cost of the Szabo equation is expensive. The development of the fast algorithm is an important issue.

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