Regularized meshless method for nonhomogeneous problems

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A R T I C L E   I N F O

Article history:
Received 10 June 2010
Available online 24 August 2010

Keywords:
Regularized meshless method
Dual reciprocity method
Nonhomogeneous problem

A B S T R A C T

The regularized meshless method is a novel boundary-type meshless method but by now has largely been confined to homogeneous problems. In this paper, we apply the regularized meshless method to the nonhomogeneous problems in conjunction with the dual reciprocity technique in the evaluation of the particular solution. Numerical experiments of three benchmark nonhomogeneous problems demonstrate the accuracy and efficiency of the present strategy.

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1. Introduction

In recent years, the boundary-type meshless methods have attracted a growing number of mathematicians and engineers [1–6]. These methods require neither domain nor boundary mesh discretization which costs much time in dealing with large-scale problems in using traditional numerical methods, such as the finite difference, finite element, and boundary element methods. The method of fundamental solutions (MFS) [7,8], the boundary knot method (BKM) [9–11], and the regularized meshless method (RMM) [12–14] are typical boundary-type meshless methods.

For the above-mentioned three methods, the RMM is found to outperform than the MFS in that the fictitious boundary in the latter is completely eliminated. Especially, the full coefficient matrix of the RMM discretization equation is well-conditioned which is superior to the ones of the MFS and BKM [15]. To the best of our knowledge, the RMM [16–18] has by now been employed to solve homogeneous problems but no report is available for its solution of nonhomogeneous problems in the literature.

In this study, we employ the regularized meshless method in conjunction with the dual reciprocity method (DRM) [19] to solve nonhomogeneous problems. This strategy can be viewed as a two-step method. First, the DRM is used to approximate the particular solution, and then the RMM is employed to calculate the homogeneous solution. In order to show the accuracy and validity, the proposed approach is tested to three benchmark examples.

The rest of the paper is divided into four sections. In Section 2, we briefly introduce the dual reciprocity method to evaluate the particular solutions, followed by Section 3 where we review the basic methodology of the regularized meshless method. Three numerical examples are investigated in Section 4. Section 5 draws some concluding remarks.

2. Dual reciprocity method for the particular solution

Without loss of generality, we consider the following boundary value problem:

\[ \nabla^2 u(x) = f(x) \quad \text{in} \, \Omega \]  
\[ u(x) = g(x) \quad \text{on} \, \Gamma_D \]  
\[ \frac{\partial u(x)}{\partial n} = h(x) \quad \text{on} \, \Gamma_N \]

where \( \nabla^2 \) is the Laplace operator and \( f(x) \) the source term. \( \Gamma_D \) represents the essential boundary (Dirichlet boundary) in which the potential is described as known \( g(x) \). \( \Gamma_N \) denotes the natural boundary (Neumann boundary) in which the normal derivative is determined as specified \( h(x) \), \( x \) the multidimensional independent variables in the interested domain \( \Omega \), \( \Gamma_D \) and \( \Gamma_N \) construct the whole boundary of \( \Omega \).

The solution of Eqs. (1)–(3) can be decomposed as

\[ u = u_h + u_p \]  (4)

where \( u_h \) and \( u_p \) represent the general and particular solutions of the present problem, respectively. The particular solution \( u_p \) satisfies

\[ \nabla^2 u_p(x) = f(x) \]  (5)

but does not necessarily satisfy the boundary conditions. The basic idea of the DRM is to expand the source function \( f(x) \) in terms of its values at the interpolation nodes \( s_k \) so that a particular solution of Eq. (5) can be calculated [20]. To evaluate
the particular solution, the right side of Eq. (5) is approximated by

$$f(\mathbf{x}) \approx \sum_{k=1}^{N+L} \beta_k \phi(r_k)$$

(6)

where $\beta_k$ are the unknown coefficients, $N$ and $L$ represent the numbers of boundary collocation points and interior collocation points, respectively. $r_k = ||\mathbf{x} - \mathbf{s}_j||$ denote the Euclidean distance. $\phi$ is the chosen radial basis function (RBF).

By collocating Eq. (6) at all collocation points, we have

$$
\begin{pmatrix}
\phi(\mathbf{x}_1, x_1) & \phi(\mathbf{x}_1, x_2) & \cdots & \phi(\mathbf{x}_1, x_{N+L}) \\
\phi(\mathbf{x}_2, x_1) & \phi(\mathbf{x}_2, x_2) & \cdots & \phi(\mathbf{x}_2, x_{N+L}) \\
\vdots & \vdots & \ddots & \vdots \\
\phi(\mathbf{x}_{N+L}, x_1) & \phi(\mathbf{x}_{N+L}, x_2) & \cdots & \phi(\mathbf{x}_{N+L}, x_{N+L})
\end{pmatrix}
\begin{pmatrix}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_{N+L}
\end{pmatrix} = 
\begin{pmatrix}
\phi(\mathbf{x}_1) \\
\phi(\mathbf{x}_2) \\
\vdots \\
\phi(\mathbf{x}_{N+L})
\end{pmatrix}
$$

(7)

that is

$$A_0 \beta = b$$

(8)

where $A_0$ is the interpolation matrix. The unknown coefficients $\beta_k$ are uniquely determined with Eq. (8). In the end we can get the particular solution of Eq. (5)

$$u_p = \sum_{k=1}^{N+L} \beta_k \phi(r_k)$$

(9)

where the RBF $\phi$ is related to the chosen RBF $\phi$ according to the governing equations. Under this condition, we need to approximate the nonhomogeneous terms via the selected RBF $\phi$. The obvious scheme is that the corresponding approximate particular solutions is determined by the integrating differential operator of the governing equation of interest. However, unless the chosen RBF $\phi$ and the governing equations are very simple, it is difficult to get the integrating terms [21]. In order to avoid this difficulty, we use an inverse sequence solution procedure. Namely, the approximate particular solution $\phi$ is chosen at first, and then we evaluate the corresponding RBF $\phi$ through substituting the specified $\phi$ into the corresponding differential operator. It is difficult to prove this procedure although it is well documented for many problems [21].

In this study, the multiquadrics (MQ) is chosen

$$\phi(r_k) = (r_k^2 + c^2)^{3/2}$$

(10)

where $c$ is the shape parameter [22]. The corresponding RBF is

$$\phi(r_k) = 60(r_k^2 + c^2)^3 + \frac{3r_k^2}{\sqrt{(r_k^2 + c^2)^2}}$$

(11)

Finally, we get particular solution at any points with coefficients $\beta_k$. For more details, we refer readers to [23].

On the other hand, the general solution $u_h$ satisfies

$$\nabla^2 u_h(\mathbf{x}) = 0 \quad \text{in} \quad \Omega$$

(12)

$$u_h(\mathbf{x}) = g(\mathbf{x}) - u_p(\mathbf{x}) \quad \text{on} \quad \Gamma_D$$

(13)

$$\frac{\partial u_h(\mathbf{x})}{\partial n} = h(\mathbf{x}) - \frac{\partial u_p(\mathbf{x})}{\partial n} \quad \text{on} \quad \Gamma_N$$

(14)

The homogeneous solution of nonhomogeneous problems is solved by the RMM which is illustrated below.

### 3. RMM for the homogeneous problems

In this section, we will show how the RMM is used to approximate the general solution of Eqs. (12)–(14) [12,13]. By using the RBF technique, the solution for Eqs. (12)–(14) at the node $\mathbf{x}_i$ can be approximated in terms of the coefficient $x_j$ and the source points $\mathbf{s}_j$ as

$$u_h(\mathbf{x}_i) = \sum_{j=1}^{N} x_j A(\mathbf{x}_i, \mathbf{s}_j)$$

(15)

$$\frac{\partial u_h(\mathbf{x}_i)}{\partial n} = \sum_{j=1}^{N} x_j B(\mathbf{x}_i, \mathbf{s}_j)$$

(16)

where $A(\mathbf{x}, \mathbf{s}_j)$ are the RBF functions, $\mathbf{s}_j$ the jth source point on the boundary, $N$ the total source point number, and $x_j$ the unknown coefficients, $B(\mathbf{x}, \mathbf{s}_j) = \partial A(\mathbf{x}, \mathbf{s}_j) / \partial n$. In the RMM, the chosen RBF functions for the interior problems are the double layer potentials from potential as follows [13]:

$$A(\mathbf{x}, \mathbf{s}_j) = -\frac{n_j y_j}{r_j^3}$$

(17)

$$B(\mathbf{x}, \mathbf{s}_j) = 2\frac{n_j y_j \pi_i}{r_j^3} - \frac{n_i \pi_j}{r_j^3}$$

(18)

where $r_j = ||\mathbf{x} - \mathbf{s}_j||$, $n_j$ is the kth component of the outward normal vector at $\mathbf{s}_j$, $\pi_j$ is the kth component of the outward normal vector at $\mathbf{x}_i$, and $y_j = \mathbf{x}_i - \mathbf{s}_j$.

It is noted that when the nodes $\mathbf{x}_i$ approach to the nodes $\mathbf{s}_j$, the double layer potentials in Eqs. (17)–(18) will be either singular or hypersingular. To overcome this troublesome issue, the desingularization technique of subtracting and adding-back is employed in the RMM to derive the diagonal elements. It is based on the discretization of the reduced null-fields equations [12,24]:

$$\int_{\Gamma} \overline{A}(\mathbf{x}, \mathbf{s}_i) \, d\Gamma(s) = 0, \quad \mathbf{x}_i \in \overline{\Omega}$$

(19)

$$\int_{\Gamma} \overline{B}(\mathbf{x}, \mathbf{s}_i) \, d\Gamma(s) = 0, \quad \mathbf{x}_i \in \overline{\Omega}$$

(20)

where $\overline{A}$ and $\overline{B}$ are generated by the opposite normal direction related to $A$ and $B$, and $\overline{\Omega}$ is the exterior domain of the $\Omega$. According to the dependence of the normal vectors for these kernel functions [12], the relationships are

$$A(\mathbf{x}, \mathbf{s}_i) = -\overline{A}(\mathbf{x}, \mathbf{s}_i), \quad i \neq j$$

$$A(\mathbf{x}, \mathbf{s}_i) = +\overline{A}(\mathbf{x}, \mathbf{s}_i), \quad i = j$$

(21)

$$B(\mathbf{x}, \mathbf{s}_i) = \overline{B}(\mathbf{x}, \mathbf{s}_i), \quad i \neq j$$

$$B(\mathbf{x}, \mathbf{s}_i) = \overline{B}(\mathbf{x}, \mathbf{s}_i), \quad i = j$$

(22)

In the RMM, Eqs. (19) and (20) are discretized as

$$\sum_{j=1}^{N} \overline{A}(\mathbf{x}, \mathbf{s}_j)|L_j| = 0$$

(23)

$$\sum_{j=1}^{N} \overline{B}(\mathbf{x}, \mathbf{s}_j)|L_j| = 0$$

(24)

where $|L_j|$ is the half distance between the $(j-1)$st and $(j+1)$st source point, when the distribution of the points are uniform, we can reduce Eq. (23) to the following:
Then we have

\[ u_j(x_i) = \sum_{j=1}^{N} z_j A(x_i, s_j) - \sum_{j=1}^{N} z_j \tilde{A}(x_i, s_j) \]

\[ = \sum_{j=1}^{N} z_j A(x_i, s_j) + \sum_{j=1}^{N} z_j \tilde{A}(x_i, s_j) \]

\[ + \left[ \sum_{j=1}^{N} \tilde{A}(x_i, s_j) - A(x_i, s_j) \right] z_i \]

\[ \frac{\partial u_j(x_i)}{\partial N} = \sum_{j=1}^{N} z_j B(x_i, s_j) - \sum_{j=1}^{N} z_j \tilde{B}(x_i, s_j) \]

\[ = \sum_{j=1}^{N-1} z_j B(x_i, s_j) + \sum_{j=1}^{N} z_j \tilde{B}(x_i, s_j) \]

\[ - \left[ \sum_{j=1}^{N} \tilde{B}(x_i, s_j) - B(x_i, s_j) \right] z_i \]

Finally, we can get the diagonal elements of RMM given by

\[ A(x_i, s_j) = \sum_{j=1}^{N} A(x_i, s_j), \quad x_i \in \Gamma_D \]  

(25)

\[ B(x_i, s_j) = - \sum_{j=1}^{N} B(x_i, s_j), \quad x_i \in \Gamma_N \]  

(26)

when the source points are not uniformly distributed, then the \(|L(j)|\) in Eq. (23) is not a constant for different \(j\). for this case, the diagonal term of RMM is still under intensive study, some recent work see Ref. [25]. In this paper, we use the diagonal elements generated by Eqs. (25)–(26). By collocating \(N\) observation points to satisfy the boundary conditions equations (13)–(14), we can get the final system of Eqs. (15)–(16). After the coefficients of Eqs. (15)–(16) is calculated, the general solution of the interest domain can be approximated by the summation of the proposed RBF with the coefficients \(z_j\) namely

\[ u_j(x) = \sum_{j=1}^{N} z_j A(x, s_j), \quad x \in \Omega \]  

(27)

4. Numerical results and discussions

This section is to test the accuracy and validity of the above-proposed method to the following three nonhomogeneous problems. The relative average error \((L_2\text{ relative error})\) in the following figures is defined as below [10]:

\[ \text{RMSE} = \sqrt{\frac{1}{N_j} \sum_{j=1}^{N_j} \left( \frac{|u(x_i) - \tilde{u}(x_i)|}{u(x_i)} \right)^2}, \]  

for \(|u(x_i)| \geq 10^{-3}\) and

\[ \text{RMSE} = \frac{1}{N_j} \sum_{j=1}^{N_j} \left( \frac{|u(x_i) - \tilde{u}(x_i)|}{u(x_i)} \right)^2, \]  

for \(|u(x_i)| < 10^{-3}\), where \(j\) is the index of the test points, and \(N_j\) represents the total number of it. \(u(x_i)\) and \(\tilde{u}(x_i)\) denote the exact and the numerical solutions, respectively.

4.1. Case 1: circular domain case

We consider the nonhomogeneous problem on \(\Omega = \{(x, y) | -1 < x < 1, 0 < y, x^2 + y^2 < 1\}\) under two different conditions:

Case 1.1: Only Dirichlet boundary is considered in this case. The analytical solution is \(u(x, y) = x^2y^2 - 2xy^2 + 10\) for the nonhomogeneous problem with source term \(f(x, y) = 6xy\).

Case 1.2: Here, we consider the mixed boundary conditions with \(\Omega = \{(x, y) | -1 < x < 1, 0 < y, x^2 + y^2 < 1\}\) Neumann boundary and the rest Dirichlet boundary. The analytical solution is \(u(x, y) = x^2y^2 - 2xy^2 + 10\) for the nonhomogeneous problem with source term \(f(x, y) = 6xy\).

It is seen from Fig. 1 that the shape parameter \(c\) in the MQ has little effect on the accuracy of the proposed method. Here, we chose the shape number \(c=2\) for these two cases. Figs. 2 and 3 depict the relative average error against the number of the boundary points, which show the solution converges very well. This validates the accuracy of the proposed two-step methods for the nonhomogeneous problems. However, we can also find that the solution accuracy is not high \(\text{RMSE}=10^{-2}\). One of the merits of the RMM is that the condition number of the matrix generated by RMM (in this case is less than \(10^2\)) is far smaller than the one generated by the MFS or BKM.

4.2. Case 2: square domain case with Dirichlet boundary only

This case is concerned with the nonhomogeneous problem on \(\Omega = \{(x, y) | 0 < x < 1, 0 < y < 1\}\) under Dirichlet boundary only.
The analytical solution is $u(x,y) = e^{xy}$ and the source term is given by $f(x,y) = e^{xy}$. In order to show the irrelevance of the MQ shape parameter to the accuracy of this combined method, the shape parameter in this case is chosen to be $c = 0.01$. Fig. 4 depicts the curve of the relative average error against the boundary point number $N$. We can see that the approximate solution converges very well.

4.3. Case 3: complex domain case with Dirichlet boundary only

The last case is to examine the proposed method under a complex domain, as shown in Fig. 5, only Dirichlet boundary condition is considered. The analytical solution for this case is

$$u(x,y) = e^{0.5x\cos(0.4y)}$$

(30)

with the source term $f(x,y) = 0.09e^{0.5x\cos(0.4y)}$. Because the BKM has difficulty in dealing with Laplace problems, the method of fundamental solutions is considered for comparison. The source points are distributed on a circle surrounding the solution domain, and the radius $R = 3$ is chosen [7]. The shape parameter in MQ is chosen to be $c = 2$.

![Fig. 3. The relative average error against the number of boundary points for case 1.2.](image)

![Fig. 4. The relative average error against the number of boundary points for case 2.](image)

![Fig. 5. Configuration of 2D complex domain.](image)

The relative average error, the DRM condition number and the condition number of the coefficient matrix generated by MFS and RMM are displayed in Tables 1 and 2, respectively. We observe that the DRM condition number of the RMM is a little lower than the one in the MFS, and the condition number of the RMM is smaller than the one generated by MFS. The accuracy of the RMM is a little lower than the MFS, but with the increasing of boundary point number, the condition number of the RMM coefficient matrix remains small, while the condition number of the MFS has a large increasing rate which may contribute to non-stable numerical results [11].

<table>
<thead>
<tr>
<th>$N$</th>
<th>Cond(DRM)</th>
<th>Cond(MFS)</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>$1.69 \times 10^4$</td>
<td>$3.32 \times 10^9$</td>
<td>$8.7 \times 10^{-2}$</td>
</tr>
<tr>
<td>20</td>
<td>$2.45 \times 10^5$</td>
<td>$3.31 \times 10^9$</td>
<td>$7.8 \times 10^{-2}$</td>
</tr>
<tr>
<td>25</td>
<td>$3.41 \times 10^6$</td>
<td>$3.33 \times 10^9$</td>
<td>$7.7 \times 10^{-2}$</td>
</tr>
<tr>
<td>30</td>
<td>$2.77 \times 10^7$</td>
<td>$3.36 \times 10^9$</td>
<td>$7.7 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

**Table 1**

MFS-DRM for case 3.

<table>
<thead>
<tr>
<th>$N$</th>
<th>Cond(DRM)</th>
<th>Cond(RMM)</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>$5.16 \times 10^4$</td>
<td>$1.47 \times 10^3$</td>
<td>$2.30 \times 10^{-4}$</td>
</tr>
<tr>
<td>20</td>
<td>$1.76 \times 10^6$</td>
<td>$1.73 \times 10^4$</td>
<td>$2.94 \times 10^{-4}$</td>
</tr>
<tr>
<td>25</td>
<td>$4.47 \times 10^7$</td>
<td>$1.88 \times 10^5$</td>
<td>$2.99 \times 10^{-4}$</td>
</tr>
<tr>
<td>30</td>
<td>$1.47 \times 10^8$</td>
<td>$2.13 \times 10^6$</td>
<td>$3.00 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

**Table 2**

RMM-DRM for case 3.

5. Conclusion

This paper develops the regularized meshless method for the nonhomogeneous problem. To investigate its applicability, we examine the three benchmark numerical examples. Numerical results show that this combined RMM-DRM strategy works well for the tested nonhomogeneous problems and numerical results converges very well against the boundary point numbers. The condition number of the RMM is relatively far smaller than those of the BKM and the MFS. The present strategy is mathematically
simple, truly meshless, integration-free, and easy-to-program. To our best knowledge, this is the first attempt to solve nonhomogeneous problems using the novel RMM.

Acknowledgement

The work described in this paper was supported by a research project funded by the Colleges and Universities in Jiangsu Province plans to graduate research and innovation (Project no. CX09B–154Z) and the Science Foundation of Hohai University (Project no. 409251).

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