FINITE DIFFERENCE SCHEMES FOR VARIABLE-ORDER TIME FRACTIONAL DIFFUSION EQUATION

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Variable-order fractional diffusion equation model is a recently developed and promising approach to characterize time-dependent or concentration-dependent anomalous diffusion, or diffusion process in inhomogeneous porous media. To further study the properties of variable-order time fractional subdiffusion equation models, efficient numerical schemes are urgently needed. This paper investigates numerical schemes for variable-order time fractional diffusion equations in a finite domain. Three finite difference schemes including the explicit scheme, the implicit scheme and the Crank-Nicholson scheme are studied. Stability conditions for these three schemes are provided and proved via the Fourier method. Rigorous convergence analysis is also performed. Two numerical examples are offered to verify the theoretical analysis of the above three schemes and illustrate the effectiveness of suggested schemes. The numerical results illustrate that the implicit scheme and the Crank-Nicholson scheme can achieve high accuracy compared with the explicit scheme, and the Crank-Nicholson scheme claims highest accuracy in most situations. Moreover, some properties of variable-order time fractional diffusion equation model are also shown by numerical simulations.

Keywords: Variable-order fractional derivative; anomalous diffusion; explicit scheme; implicit scheme; Crank–Nicholson scheme; Fourier method.

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1. Introduction

Fractional calculus has received extensive attention in the fields of anomalous diffusion [Metzler & Klafter, 2000; Zaslavsky, 2002; Bao, 2003], viscoelasticity [Bagley & Torvik, 1983; Schießel et al., 1995], control [Podlubny, 1999; Chen & Moore, 2002], turbulence [del-Castillo-Negrete et al., 2004; Chen, 2006], chaos [Li & Peng, 2004], etc. Among these, the anomalous diffusion modeled by fractional calculus is one of the most significant fields in fractional calculus application. Anomalous diffusion phenomena are extensively observed in physics, chemistry and biology fields [Solomon et al., 1993; Bhalekar et al., 2011; Magin et al., 2008; Magin, 2006]. To characterize anomalous diffusion phenomena, constant-order fractional diffusion equations are introduced and have received tremendous success. However, it has been found that the constant-order fractional diffusion equations are not capable of characterizing some complex diffusion processes, for instance, diffusion process in inhomogeneous or heterogeneous medium [Chechkin et al., 2005]. In addition, when we consider diffusion process in porous medium, if the medium structure or external field changes with time, in this situation, the constant-order fractional diffusion equation model cannot be used to well characterize such phenomena [Santamaria et al., 2006; Sun et al., 2009].

Still in some biology diffusion processes, the concentration of particles will determine the diffusion pattern [Sun et al., 2009; de Azevedo et al., 2006]. To solve the above problems, the variable-order (VO) fractional diffusion equation models have been suggested for use [Chechkin et al., 2005; Sun et al., 2011; Umarov & Steinberg, 2009].

The pioneering work of VO operator can be traced to Samko et al. by introducing the variable-order integration and Riemann–Liouville derivative in 1993 [Samko & Ross, 1995; Ross & Samko, 1995]. It has been recognized as a powerful modeling approach in the fields of viscoelasticity [Coimbra, 2003], viscoelastic deformation [Ingman & Susidnitsky, 2005], viscous fluid [Pedro et al., 2008], anomalous diffusion [Kobele et al., 2003], etc. For example, Chechkin et al. obtained the VO time-fractional diffusion equation from the continuous time random walk (CTRW) scheme, which promoted the modeling and application study of the VO fractional equation model in anomalous diffusion and related fields [Chechkin et al., 2005]. These current applications of VO fractional differential equations, and many others that may well be proposed in the near future, make it imperative to search for methods of exact or numerical solution [Lim & Teo, 2008; Yuste & Acedo, 2005]. In the general cases, compared with its counterpart constant-order one, the analytical solution of the VO fractional differential equation is hard to obtain. Therefore, the numerical solution is a more practical way to investigate the VO evolutionary equation. There are two types of VO fractional differential definitions. One considers the derivative order has no memory related to past derivative order values, the other definition focuses on the cases in which the derivative order has memory related to past derivative order values [Sun et al., 2009; Coimbra, 2003].

The VO fractional derivative definition which corresponds to this property can be stated as follows

\[ C_0^\alpha D_t^{\alpha(t)} f(t) = \frac{1}{\Gamma(1-\alpha(t))} \int_0^t \frac{f(\tau)}{(t-\tau)^\alpha(t)} d\tau, \quad 0 < \alpha(t) \leq 1, \quad (1) \]

where \( C_0^\alpha D_t^{\alpha(t)} \) is the VO fractional derivative in Caputo sense, \( \alpha(t) \) derivative order, 0+ initial value, \( \Gamma() \) Gamma function. The finite difference schemes for constant-order time or space fractional diffusion equations have been widely studied [Lynch et al., 2003; Zhang, 2009; Tadjeran et al., 2006; Meerschaert & Tadjeran, 2004; Deng, 2007; Li et al., 2011]. For constant-order time fractional diffusion equations, Chen et al. proposed implicit difference approximation scheme [Chen et al., 2007], Yuste et al. introduced weighted average finite difference methods [Yuste, 2006]. Podlubny et al. proposed the matrix approach for fractional diffusion equations [Podlubny et al., 2009] and Hanert proposed a flexible numerical scheme for the discretization of the space-time fractional diffusion equation [Hanert, 2011]. Recently, Zhuang et al. considered the numerical schemes for VO space fractional advection-dispersion equation [Zhuang et al., 2009]. Lin et al. investigated the explicit scheme for VO nonlinear space fractional diffusion equation [Lin et al., 2009]. Then, in this study, we will consider the finite difference schemes for a series of VO time fractional diffusion equations. Analysis and comparison of stability, convergence and...
accuracy for three finite difference schemes including the explicit scheme, the implicit scheme and the Crank–Nicolson scheme are presented.

The structure of this study is constructed as follows. The construction of three finite difference schemes for VO time fractional diffusion equation is presented in Sec. 2. In Sec. 3, we offer the detailed stability analysis of the three schemes. The convergence of the three schemes is discussed in Sec. 4. Two numerical examples are offered to demonstrate the efficiency of three schemes in Sec. 5. We complete this paper by some remarks in Sec. 6.

2. Three Finite Difference Schemes

We employ the explicit scheme, the implicit scheme and the Crank–Nicholson scheme to perform numerical simulation in this section. The considered VO time fractional diffusion equation is stated as follows

\[
\begin{aligned}
&\left\{ \begin{array}{l}
C_0 D_t^{\alpha(x,t)} u(x,t) = K \frac{\partial^2 u(x,t)}{\partial x^2} + q(x,t), 
& x \in (0, L), \ t \in (0, T),

&u(x, 0) = f(x),

&u(0, t) = g(t); \ u(L, t) = h(t),
\end{array} \right.
\end{aligned}
\]

where \( K > 0 \) is a generalized diffusion coefficient, \( u(x, t) \) is concentration, mass or other physical quantities of interest, \( q(x, t) \) is a source term, \( f(x) \) is the initial condition, \( g(t) \) and \( h(t) \) are the boundary conditions, \( L \) and \( T \) are the lengths of space and time, \( \alpha(x, t) \in (0, 1] \) is the Caputo-type VO fractional derivative order, it can be a function of time, space or other independent variables.

Before performing the numerical methods, we assume Eq. (2) has a unique and sufficiently smooth solution. To establish the numerical schemes for the above equation, we let \( x_l = l h, 0 \leq l \leq M, M h = L, \ t_k = k \tau, 0 \leq k \leq N, N \tau = T, h \) is the space step length and \( \tau \) is the time step size, \( M \) and \( N \) are grid points.

2.1. Explicit difference scheme

The finite difference approximation for the second order spatial derivative can be stated as follows

\[
\frac{\partial^2 u(x_l, t_k)}{\partial x^2} = \frac{u(x_{l+1}, t_k) - 2u(x_l, t_k) + u(x_{l-1}, t_k)}{h^2} + O(h^2).
\]

The discretization of the Caputo-type VO time fractional derivative can be stated as follows [Zhuang & Liu, 2006; Murio, 2008]

\[
\frac{\partial^{2+\alpha} u(x_l, t_{k+1})}{\partial t^{2+\alpha}} = \frac{\tau^{-\alpha^{k+1}}}{\Gamma(2 - \alpha^{k+1})} \left\{ u(x_l, t_{k+1}) - u(x_l, t_k) \right. \\
+ \sum_{j=1}^{k} [u(x_l, t_{k+1-j}) - u(x_l, t_{k-j})][(j + 1)^{1-\alpha^{k+1}} - j^{1-\alpha^{k+1}}] \bigg\} + O(\tau).
\]

Therefore, the explicit scheme can be written as

\[
u_{l}^{k+1} = r_{l}^{k+1} u_{l}^{k+1} + (1 - 2r_{l}^{k+1}) u_{l}^{k} + r_{l}^{k+1} u_{l}^{k-1} - \frac{k}{\Gamma(2 - \alpha^{k+1})} \sum_{j=1}^{k} [u_{l}^{k+1-j} - u_{l}^{k-j}] [j^{1-\alpha^{k+1}} - j^{1-\alpha^{k+1}}] + \tau \Gamma(2 - \alpha^{k+1}) q_{l}^{k+1},
\]

where \( r_{l}^{k+1} = \frac{K}{\Gamma(2 - \alpha^{k+1})} \Gamma(2 - \alpha^{k+1}) [j^{1-\alpha^{k+1}} - j^{1-\alpha^{k+1}}], j = 1, 2, \ldots, N, \) and \( l = 1, 2, \ldots, M - 1. \) It should be pointed out that the sum term on the right-hand side of (5) automatically vanishes when \( k = 0. \) Then, Eq. (5) can be rewritten as
It can be transferred into a matrix form

\[
\begin{align*}
A(1)\mathbf{u}^k & = J\mathbf{u}^0 + Q^1, \\
A(k+1)\mathbf{u}^{k+1} & = \mathbf{M}^{k+1}\mathbf{u}^k + \cdots + \mathbf{M}\mathbf{u}^1 + \mathbf{M}^k\mathbf{u}^0 + Q^{k+1},
\end{align*}
\]

where

\[
A(k) = \begin{bmatrix}
1 + 2r_1^k & -r_1^k & 0 & \cdots & 0 & 0 \\
-r_1^k & 1 + 2r_2^k & -r_2^k & \cdots & 0 & 0 \\
0 & -r_2^k & 1 + 2r_3^k & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 + 2r_{M-2}^k & -r_{M-2}^k \\
0 & 0 & 0 & \cdots & -r_{M-1}^k & 1 + 2r_{M-1}^k
\end{bmatrix}
\]

(11)
The above discretization can be rewritten in the following matrix form

\[ u^k = \begin{bmatrix} u_1^k \\ \vdots \\ u_{M-1}^k \end{bmatrix}, \quad \Delta = \begin{bmatrix} b_{j,1}^k \\ \vdots \\ b_{j,M-1}^k \end{bmatrix}, \quad (12) \quad Q^k = \begin{bmatrix} r^2 f(t_k) + \tau^2 \Gamma(2 - \alpha_1^k) \eta_1^k \\ \vdots \\ r^2 \eta_{M-2}^k \Gamma(2 - \alpha_{M-2}^k) \eta_{M-2}^k \end{bmatrix}, \]

and \( I \) is an \((M-1) \times (M-1)\) identity matrix.

2.3. Crank–Nicholson scheme

We introduce the Crank–Nicholson scheme as follows. Firstly, the discretization of second order space derivative is stated as

\[
\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{2} \left[ \frac{u(x_{i+1}, t_{k+1}) - 2u(x_i, t_{k+1}) + u(x_{i-1}, t_{k+1})}{h^2} \right] + O(h^2). \tag{15}
\]

The discretization of the Caputo-type VO time fractional derivative is the same as (4). Then, the detailed Crank–Nicholson scheme for the VO time fractional diffusion model can be stated as follows

\[
-r_i^k u_{i+1}^{k+1} + (1 + 2r_i^k) u_i^{k+1} - r_i^{k+1} u_{i-1}^{k+1} = r_i^{k+1} u_{i+1}^{k+1} + \left[ 1 - b_i^{k+1} \right] u_i^{k+1} + \sum_{j=1}^{k} \left[ b_i^{k+1-j} - u_{i}^{k-j} \right] \eta_i^{k-j}, \tag{16}
\]

where \( r_i^{k+1} = \frac{K^{k+1}}{2^{k+1} - \alpha^{k+1}} \).

It can be transformed into the following form

\[
\begin{align*}
-r_i^k u_{i+1}^{k+1} + (1 + 2r_i^k) u_i^{k+1} - r_i^{k+1} u_{i-1}^{k+1} = r_i^{k+1} u_{i+1}^{k+1} + (1 - b_i^{k+1}) u_i^{k+1} + \sum_{j=1}^{k} \left[ b_i^{k+1-j} - u_{i}^{k-j} \right] \eta_i^{k-j}, \quad k = 0, \\
=-r_i^{k+1} u_{i+1}^{k+1} + (1 + 2r_i^{k+1}) u_i^{k+1} - r_i^{k+1} u_{i-1}^{k+1} = r_i^{k+1} u_{i+1}^{k+1} + (1 - b_i^{k+1}) u_i^{k+1} + \sum_{j=1}^{k} \left[ b_i^{k+1-j} - u_{i}^{k-j} \right] \eta_i^{k-j}, \quad k \geq 1, \tag{17}
\end{align*}
\]

The above discretization can be rewritten in the following matrix form

\[
\begin{align*}
A(1) u^1 = B(1) u^0 + Q^{1/2}, \\
A(k+1) u^{k+1} = B(k+1) u^k + \eta_1^{k+1} u^k + \eta_2^{k+1} u^k + \cdots + \eta_{M-1}^{k+1} u^k + \eta_M^{k+1} u^k + Q^{k+1/2}, \quad k \geq 1, \tag{18}
\end{align*}
\]
is defined as follows

\[ l = 1, 2, \ldots, N. \]

The expressions of \( u^k \), \( U_j^{k+1} \) and \( \Phi_j^{k+1} \) are the same as those in the implicit scheme.

3. Stability Analysis of Three Difference Schemes

In this section, we will analyze the stability conditions of the explicit scheme, the implicit scheme and the Crank–Nicholson scheme.

Let \( \rho^k = u^k - U^k \), where \( U^k \) represents the approximate solution at \((x_i, t_k), (k = 1, 2, \ldots, N; l = 1, 2, \ldots, M - 1)\), and

\[ \rho^k = [\rho^1, \rho^2, \ldots, \rho^M]^{T}. \]

Next, we will analyze the stability via the Fourier method [Chen et al., 2007; Langlands & Henry, 2005], the expression of \( \rho^k \) \((k = 1, 2, \ldots, N)\) is defined as follows

\[ \rho^k = \begin{cases} \rho^l, & \text{if } x_l - \frac{h}{2} < x \leq x_l + \frac{h}{2}, \\ 0, & \text{if } 0 \leq x < \frac{h}{2} \text{ or } L - \frac{h}{2} < x \leq L, \end{cases} \]

then, \( \rho^k(x) \) can be expanded in a Fourier series:

\[ \rho^k(x) = \sum_{m=-\infty}^{\infty} \delta_k(m)e^{2\pi i m x / L}, \quad (k = 1, 2, \ldots, N), \]

where

\[ \delta_k(m) = \frac{1}{L} \int_{0}^{L} \rho^k(x)e^{-2\pi i m x/L}dx. \]
It can be proved that [Chen et al., 2007]
\[
\|\rho_t\|^2 = \sum_{n=-\infty}^{\infty} |\delta_k(n)|^2.
\]
(24)

Proposition 1. The coefficients \( r_k^j \) and \( \delta_k^j \) have the following properties,

(1) \( r_k^j > 0 \), \( 0 < b_{j+1}^k < b_j^k \leq 1 \),
\[ \forall l = 1, 2, \ldots, M; \quad \forall j, k = 1, 2, \ldots, N, \]
(2) \( 0 < \delta_k^j < 1 \), \[ \sum_{j=0}^{k-1} d_{j+1}^k = 1 - b_k^k. \]

3.1. Stability analysis of the explicit scheme

First, we can get the following roundoff error equation from the explicit scheme Eq. (5)
\[
\rho_{l+1}^k = r_n^1 \rho_{l+1}^n + \left( 1 - b_n^1 + 2r_n^1 \right) \rho_n^l
+ r_n^1 \rho_{l+1}^n + \sum_{j=1}^{k-1} \rho_{l+1}^j b_j^{k+1} + b_j^{k+1} \rho_{l+1}^n,
\]
\[ k = 1, 2, \ldots, N - 1. \]
(26)

We suppose \( \rho_n^l \) in Eq. (26) has the following form
\[
\rho_n^l = \delta_n \sin \sigma l,
\]
(27)

where \( \sigma \) is a real spatial wave number.

Substituting it into Eq. (26), we can get
\[
\delta_{k+1} = \left[ 1 - b_l^{k+1} - 4r_l^{k+1} \sin^2 \left( \frac{\sigma h}{2} \right) \right] \delta_k
+ \sum_{j=1}^{k-1} d_{j+1}^k \delta_{k-j} + b_k^{k+1} \delta_0,
\]
\[ k = 0, 1, 2, \ldots, N - 1. \]
(28)

Lemma 1. Suppose that \( \delta_k \) \( (k = 1, 2, \ldots, N - 1) \) is the solution of Eq. (28), and for \( \forall (l, k), r_l^n \leq \frac{1}{4} \left( 1 - b_l^1 \right), (l = 1, 2, \ldots, M; k = 1, 2, \ldots, N), \) then
\[ |\delta_k| \leq |\delta_0|, \] holds for \( k = 0, 1, 2, \ldots, N - 1. \)

Proof. Here, we employ the mathematical induction to perform the proof. From (7) and (28), for \( \forall l = 1, 2, \ldots, M - 1, \) when \( k = 0 \) in (28), we can get
\[
\delta_1 = \left[ 1 - 4r_1^1 \sin^2 \left( \frac{\sigma h}{2} \right) \right] \delta_0.
\]
Noticing that \( r_1^1 > 0 \) and \( r_1^n \leq \frac{1}{4} \), we get
\[ |\delta_1| \leq |\delta_0|. \]

Next, we let \( |\delta_n| \leq |\delta_0| \) (\( n = 2, 3, \ldots, k \)). From Eq. (28), we can get
\[
|\delta_{k+1}| \leq \left[ 1 - b_k^{k+1} - 4r_k^{k+1} \sin^2 \left( \frac{\sigma h}{2} \right) \right] |\delta_k|
+ \sum_{j=1}^{k-1} d_{j+1}^k |\delta_{k-j}| + b_k^{k+1} |\delta_0|,
\]
\[ \leq 1 - b_k^{k+1} - 4r_k^{k+1} \sin^2 \left( \frac{\sigma h}{2} \right) + \sum_{j=1}^{k-1} d_{j+1}^k + b_k^{k+1} |\delta_0|,
\]
since \( d_j^{k+1} = 1 - b_j^{k+1} \) and Proposition 1,
\[ |\delta_{k+1}| \leq \left[ 1 - 4r_k^{k+1} \sin^2 \left( \frac{\sigma h}{2} \right) \right] |\delta_k|,
\]
Then, we obtain
\[ |\delta_{k+1}| \leq \left[ 1 - 4r_k^{k+1} \sin^2 \left( \frac{\sigma h}{2} \right) \right] |\delta_k| \leq |\delta_0|. \]

This completes the proof. \( \blacksquare \)

Theorem 1. The explicit difference scheme (5) is stable under the condition that \( \forall (l, k), r_l^n \leq \frac{1}{4} (1 - b_l^1), (l = 1, 2, \ldots, M; k = 1, 2, \ldots, N), \)

Proof. From Eq. (24) and Lemma 1, we can obtain
\[ \|\rho_n^l\|_2 \leq \|\rho_n^l\|_2, \text{ } k = 1, 2, \ldots, N - 1, \]
which proves that the explicit scheme is stable. \( \blacksquare \)

Remark. From Theorem 1, we know that the stability condition of explicit scheme is dependent on the evolution trajectory of \( \alpha(x, t) \). Hereby, the stability condition may change with the time evolution or the space location.
3.2. Stability analysis of the implicit scheme

For the implicit scheme, we can get the following roundoff error equation from (9)

\[
(1 + 2r^k_{i+1})p_{l,i+1}^k - r^k_{i+1}p_{l,i+1}^k - r^k_{i}p_{l,i+1}^k = \sum_{j=0}^{k-1} \rho_0^{k-j} d_{j,i+1}^k .
\]  

(30)

We suppose that the solution of Eq. (30) has the following form

\[
\rho_i^k = \delta_k e^{i\omega t} .
\]  

(31)

Substituting it into Eq. (30), we get

\[
\begin{cases}
(1 + 4r^k_{i} \sin^2 \left( \frac{\sigma h}{2} \right) ) \delta_{i} = \delta_0 , & k = 0, \\
(1 + 4r^k_{i+1} \sin^2 \left( \frac{\sigma h}{2} \right) ) \delta_{i+1} = \sum_{j=0}^{k-1} d_{j,i+1}^k \delta_{i-j} + b_i^{k+1} \delta_0 , & k = 1, 2, \ldots, N-1.
\end{cases}
\]  

(32)

The above equation can be rewritten as

\[
|\delta_{i+1}| \leq \sum_{j=0}^{k-1} |d_{j,i+1}^k| |\delta_{i-j}| + \frac{b_i^{k+1} \delta_0}{1 + 4r^k_{i+1} \sin^2 \left( \frac{\sigma h}{2} \right)} \leq \left[ \sum_{j=0}^{k-1} |d_{j,i+1}^k| \right] |\delta_0| + \frac{b_i^{k+1} \delta_0}{1 + 4r^k_{i+1} \sin^2 \left( \frac{\sigma h}{2} \right)} |\delta_0| , \quad k = 1, 2, \ldots, N-1.
\]  

(34)

Due to Proposition 1, we get

\[
|\delta_{i+1}| \leq \frac{b_i^{k+1} \delta_0}{1 + 4r^k_{i+1} \sin^2 \left( \frac{\sigma h}{2} \right)} |\delta_0| .
\]

Since \( b_i^{k+1} = 1 \), then

\[
|\delta_{i+1}| \leq \frac{1}{1 + 4r^k_{i+1} \sin^2 \left( \frac{\sigma h}{2} \right)} |\delta_0| \leq |\delta_0| .
\]

This ends the proof.

Theorem 2. The implicit difference scheme (10)–(14) is unconditionally stable.

Proof. From Eq. (24) and Lemma 2, we can obtain

\[
\|\rho^h\|_2 \leq \|\rho^h\|_2 , \quad k = 1, 2, \ldots, N-1,
\]  

(35)

which shows that the implicit scheme is unconditionally stable.
3.3. Stability analysis of the Crank–Nicholson scheme

The roundoff error equation corresponding to (17) can be stated as follows

\[(1 + 2r_i^{k+1})\rho_i^{k+1} - r_i^k \rho_{i+1}^k - r_i^k \rho_{i-1}^k = \rho_i^k + (1 - 2r_i^{k+1} - b_i^{k+1})\rho_i^k + r_i^k \rho_{i+1}^k - \sum_{j=1}^{k-1} d_{i+1}^{k+1} \rho_{i-j}^k + b_i^{k+1} \rho_{i+1}^k, \quad k \geq 1.\]  

(36)

In the same way, we suppose that the solution of Eq. (36) has the following form

\[\rho_i^k = \delta_i e^{\rho t_k}.\]  

(37)

Substituting it into Eq. (36), gives

\[\begin{cases}
1 + 4r_i^{k+1} \sin^2\left(\frac{\sigma h}{2}\right) \delta_k = 1 - 4r_i^{k+1} \sin^2\left(\frac{\sigma h}{2}\right) \delta_0, & k = 0, \\
1 + 4r_i^{k+1} \sin^2\left(\frac{\sigma h}{2}\right) \delta_{k+1} = 1 - b_i^{k+1} - 4r_i^{k+1} \sin^2\left(\frac{\sigma h}{2}\right) \delta_k + \sum_{j=1}^{k-1} d_{i+1}^{k+1} \delta_{k-j} + b_i^{k+1} \delta_0, & k = 1, 2, \ldots, N - 1.
\end{cases}\]  

(38)

The above equation can be rewritten as

\[\delta_{k+1} = \frac{1 - b_i^{k+1} - 4r_i^{k+1} \sin^2\left(\frac{\sigma h}{2}\right) \delta_k + \sum_{j=1}^{k-1} d_{i+1}^{k+1} \delta_{k-j} + b_i^{k+1} \delta_0}{1 + 4r_i^{k+1} \sin^2\left(\frac{\sigma h}{2}\right)}.\]  

(39)

Lemma 3. Suppose that \(\delta_k (k = 1, 2, \ldots, N - 1)\) is the solution of Eq. (39), then, we have

\[|\delta_k| \leq |\delta_0|, \quad k = 1, 2, \ldots, N - 1.\]

Proof. Again, we use the mathematical induction to perform the proof process. From Eqs. (17) and (39), for \(\forall l (l = 1, 2, \ldots, M - 1)\), when \(k = 0\) in Eq. (38), we get

\[\delta_1 = \frac{1 - 4r_l^1 \sin^2\left(\frac{\sigma h}{2}\right) \delta_0}{1 + 4r_l^1 \sin^2\left(\frac{\sigma h}{2}\right)} \delta_0.\]

Noticing that \(r_l^1 > 0\), we get

\[|\delta_1| \leq |\delta_0|.

Next, we let \(|\delta_n| \leq |\delta_0| (n = 2, 3, \ldots, k)\). From Eq. (33), we get

\[|\delta_{k+1}| = \frac{1 - b_i^{k+1} - 4r_i^{k+1} \sin^2\left(\frac{\sigma h}{2}\right) \delta_k + \sum_{j=1}^{k-1} d_{i+1}^{k+1} \delta_{k-j} + b_i^{k+1} \delta_0}{1 + 4r_i^{k+1} \sin^2\left(\frac{\sigma h}{2}\right)} \delta_k + \sum_{j=1}^{k-1} d_{i+1}^{k+1} \delta_{k-j} + b_i^{k+1} \delta_0}
\]

\[= \frac{-4r_i^{k+1} \sin^2\left(\frac{\sigma h}{2}\right) \delta_k + \sum_{j=1}^{k-1} d_{i+1}^{k+1} \delta_{k-j} + b_i^{k+1} \delta_0}{1 + 4r_i^{k+1} \sin^2\left(\frac{\sigma h}{2}\right)} \delta_k + \sum_{j=1}^{k-1} d_{i+1}^{k+1} \delta_{k-j} + b_i^{k+1} \delta_0}.\]
In addition, we discuss some properties of the Crank-Nicholson scheme in special cases. When \( k = 0 \), the following result can be obtained
\[
\frac{\delta_{k+1}}{\delta_k} = \frac{1 - 4r_{k+1}^2 \sin^2 \left( \frac{\sigma h}{2} \right)}{1 + 4r_{k+1}^2 \sin^2 \left( \frac{\sigma h}{2} \right)} \leq 1. \quad (41)
\]
When \( k \geq 1 \), then
\[
\frac{\delta_{k+1}}{\delta_k} = \frac{1 - 4r_{k+1}^2 \sin^2 \left( \frac{\sigma h}{2} \right) - b_{k+1}^j \delta_{k+1} + \sum_{j=1}^{k} \left( b_{j+1}^{k+1} - b_{j+1}^{k+1} \delta_{j+1} \right) \delta_{j+1} + b_{k+1}^j \delta_j}{1 + 4r_{k+1}^2 \sin^2 \left( \frac{\sigma h}{2} \right)} \quad (42)
\]
In the above expression, if \( \alpha(x,t) = 1 \), then \( b_{k+1}^j = 0 \), \( j = 0, 1, \ldots, k \), and
\[
\frac{\delta_{k+1}}{\delta_k} \leq 1. \quad (43)
\]
If \( 0 < \alpha(x,t) < 1 \), we can verify the Crank-Nicholson scheme is stable via numerical simulation of Eq. (42). The numerical results for \( k = 0, 1, 2, 3, 4 \) have been shown in Fig. 1. From the observation of Fig. 1, we know that, an important feature of the Crank-Nicholson scheme is \( \frac{\delta_{k+1}}{\delta_k} \geq \frac{\delta_{k-1}}{\delta_{k-1}} \).

4. Convergence of Three Finite Difference Schemes
Assume that \( u(x_i,t_k) \), \( i = 1, 2, \ldots, M - 1; k = 1, 2, \ldots, N \) is the exact solution of Eq. (2) at \( (x_i,t_k) \),
we define \( e^k_i = u(x_i, t_k) - u^k_i \) and \( e^{k+1} = (e^k_1, \ldots, e^k_{M-1})^T \). Since \( e^0 = 0 \), we obtain the following relations for the implicit scheme from Eq. (9)

\[
\begin{align*}
(1 + 2r_i^k)e^k_i & - r_i^k e^{k+1}_{i+1} - r_i^k e^{k+1}_{i-1} = R_i^k, & k = 0, \\
(1 + 2r_i^{k+1})e^{k+1}_i & - r_i^{k+1} e^{k+1}_{i+1} - r_i^{k+1} e^{k+1}_{i-1} = \sum_{j=0}^{k-1} c_j^k d_{j+1}^{k+1} + R_i^{k+1}, & k > 0,
\end{align*}
\]

(44)

where \( R_i^{k+1} \) has the following expression

\[
R_i^{k+1} = u(x_i, t_{k+1}) - \sum_{j=0}^{k-1} d_j^{k+1} u(x_i, t_{j}) - b_i^{k+1} u(x_i, t_0) - r_i^{k+1} u(x_{i-1}, t_{k+1}) - 2u(x_i, t_{k+1}) + u(x_{i-1}, t_{k+1})).
\]

(45)

From (3) and (4), we can get

\[
\frac{\Delta \tau - \eta_i^{k+1}}{\Gamma(2 - \alpha_i^{k+1})} \left[ u(x_i, t_{k+1}) - \sum_{j=0}^{k-1} d_j^{k+1} u(x_i, t_{j}) - b_i^{k+1} u(x_i, t_0) \right] = \frac{\partial^\alpha_i^{k+1} u(x_i, t_{k+1})}{\partial t^{\alpha_i^{k+1}}} + C_1 \tau,
\]

(46)

\[
\frac{\partial u(x_i, t_{k+1})}{\partial x^2} = \frac{\partial^2 u(x_i, t_{k+1})}{\partial x^2} + C_2 h^2.
\]

Then,

\[
R_i^{k+1} = \frac{\Delta \tau - \eta_i^{k+1}}{\Gamma(2 - \alpha_i^{k+1})} \left[ \frac{\partial^\alpha_i^{k+1} u(x_i, t_{k+1})}{\partial t^{\alpha_i^{k+1}}} - \frac{\partial u(x_i, t_{k+1})}{\partial x^2} \right] + C_1 \tau \eta_i^{k+1} + C_2 h^2,
\]

(47)

which implies that

\[
R_i^{k+1} \leq C(\tau^{(1 + \alpha_i^{k+1})} + \tau^{\alpha_i^{k+1}} h^2),
\]

(48)

where \( C_1, C_2, C \) are constants. The detailed error analysis on the above schemes considering Caputo-type fractional derivative can refer to the work by Diethelm et al. [2004] and further work by Li and Tao [2009].

**Lemma 4.** \( \| e^{k+1} \|_\infty \leq C(b_i^{k+1})^{-1}(\tau^{1 + \alpha_i^{k+1}} + \tau^{\alpha_i^{k+1}} h^2), \) hold for \( k = 0, \ldots, N - 1 \), where \( \| e^k \|_\infty = \max_{1 \leq i \leq M - 1} | e_i^k | \), \( C \) is a constant and \( \alpha_i^{k+1} = \begin{cases} \min_{1 \leq i \leq M - 1} \alpha_i^{k+1}, & \text{if } \tau \leq 1, \\ \max_{1 \leq i \leq M - 1} \alpha_i^{k+1}, & \text{if } \tau > 1. \end{cases} \) (49)

**Proof.** We also employ the mathematical induction method to perform the proof process. For \( k = 0 \), we have

\[
| e_i^0 | \leq (1 + 2r_i^0)| e_i^0 | - r_i^0 | e_i^{k+1}_{i+1} | - r_i^0 | e_i^{k+1}_{i-1} |
\]
\[ \begin{aligned}
&\leq (b_0^{(k+1)} - b_k^{(k+1)} + b_k^{(k+1)}b_0^{(k+1)}) - C(\tau + \kappa + \tau^{\alpha + 1}k^2)
&\leq C(\tau + \kappa + \tau^{\alpha + 1}k^2),
\end{aligned} \tag{51} \]

which completes the proof. \( \blacksquare \)

In addition, we can obtain
\[
\lim_{k \to \infty} \frac{h^k}{k^{\alpha}} = \frac{1}{1 + \alpha},
\]

hence, we get the following result
\[
\|e^k\|_\infty \leq Ck^{\alpha}(\tau + \kappa + \tau^{\alpha + 1}k^2). \tag{53}\]

Since \( kr < T \), the following results can be obtained.

**Theorem 4.** The implicit scheme is convergent, and there exists a positive constant \( C \) such that
\[
|u_l - u(x_l, t_k)| \leq C(\tau + \kappa), \quad l = 1, 2, \ldots, M - 1, \quad k = 1, 2, \ldots, N. \tag{54}\]

With the same method, we can also prove the following theorems for the explicit scheme and the Crank-Nicholson scheme [Zhuang et al., 2009; Lin et al., 2009; Liu & Lin, 2006; Lin & Xu, 2007].

**Theorem 5.** The explicit scheme is convergent if \( \tau \leq \frac{1}{2} \left( 1 - b_1^2 \right), \quad \tau = 1, 2, \ldots, M, k = 1, 2, \ldots, N, \) and there exists a positive constant \( C \) such that
\[
|u_l - u(x_l, t_k)| \leq C(\tau + \kappa), \quad l = 1, 2, \ldots, M - 1, \quad k = 1, 2, \ldots, N. \tag{55}\]

**Theorem 6.** The Crank-Nicholson scheme is convergent, and there exists a positive constant \( C \) such that
\[
|u_l - u(x_l, t_k)| \leq C(\tau + \kappa), \quad l = 1, 2, \ldots, M - 1, \quad k = 1, 2, \ldots, N. \tag{56}\]

We should point out that, the solvability of the three schemes can also be proved [Chen et al., 2007; Cui, 2009]. So we omit the details of the proof.

5. **Numerical Examples**

**Example 1**

\[
\begin{cases}
\frac{D^\alpha u(x,t)}{Dt^\alpha} + q(x,t) = 0, & x \in (0, L), \quad t \in (0, T), \\
u(x,0) = 0, & x \in [0, L], \\
u(t,0) = u(L,t) = 0, & t \in [0, T],
\end{cases} \tag{57}
\]

where
\[
q(x,t) = \frac{2}{\Gamma(3 - \alpha(x,t))} 2^{2-\alpha(x,t)} \sin \left( \frac{\kappa x L}{T} \right) + \frac{K \tau^2 L^2}{L} \sin \left( \frac{\kappa x L}{T} \right), \tag{58}\]

\[0 < \alpha(x,t) \leq 1, \quad \forall (x,t). \quad \text{The exact analytical solution to the above equation can be stated as}
\]
\[
u(x,t) = t \sin \left( \frac{\kappa x L}{T} \right). \tag{59}\]

In these numerical examples, we assume that \( K = 0.01, \quad L = 10, \quad T = 0.5. \) First, we will compare the efficiency of the three schemes under constant-order \( \alpha(x,t) = 0.8. \) The absolute errors of the three schemes are presented in Table 1. Secondly, we change the derivative order as a function of time \( \alpha(x,t) = 0.8 + 0.2 \tau / T, \) the corresponding absolute errors of the three schemes are stated in Table 2. Thirdly, the derivative order is changed into a function which includes both time and space \( \alpha(x,t) = 0.8 + 0.2 \tau x / (LT), \) the absolute errors under this type of \( \alpha \) are then listed in Table 3. To illustrate the efficiency of the Crank-Nicholson scheme, we also present the numerical results of the VOF time fractional diffusion equation by the Crank-Nicholson scheme in Fig. 2. From comparisons of

<table>
<thead>
<tr>
<th>Time</th>
<th>Explicit Scheme</th>
<th>Implicit Scheme</th>
<th>Crank-Nicholson Scheme</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t = 0.1 )</td>
<td>0.007E-2</td>
<td>0.526E-3</td>
<td>0.465E-3</td>
</tr>
<tr>
<td>( t = 0.2 )</td>
<td>0.314E-2</td>
<td>1.080E-3</td>
<td>0.519E-3</td>
</tr>
<tr>
<td>( t = 0.3 )</td>
<td>1.114E-2</td>
<td>1.836E-3</td>
<td>1.167E-3</td>
</tr>
<tr>
<td>( t = 0.4 )</td>
<td>4.75E-2</td>
<td>2.932E-3</td>
<td>1.169E-3</td>
</tr>
<tr>
<td>( t = 0.5 )</td>
<td>4.76E-2</td>
<td>4.434E-3</td>
<td>0.628E-3</td>
</tr>
</tbody>
</table>

Table 1. Absolute errors of the explicit scheme, the implicit scheme and the Crank-Nicholson scheme for constant-order fractional diffusion equation at \( x = 5.0. \) The time step length \( \tau = 0.01, \) the space step size \( h = 0.1 \) and the time space order \( \alpha(x,t) = 0.8. \)
The numerical result of VO time fractional diffusion equation by the Crank-Nicholson scheme,

$K_{\text{Crank-Nicholson}}$. 

Fig. 2. The numerical result of VO time fractional diffusion equation at $x = 5.0$. The time step length $\tau = 0.01$, the space step size $h = 0.1$ and the VO function of time fractional derivative $\rho(x, t) = 0.8 + 0.2t/(TL)$. 

Tables 1–3, we observe that the Crank-Nicholson scheme and the Crank-Nicholson scheme for VO fractional diffusion equation at $t = 0$. 

Table 2. Absolute errors of the explicit scheme, the implicit scheme and the Crank-Nicholson scheme at $t = 0$. The time step length $\tau = 0.01$, the space step size $h = 0.1$ and the VO function of time fractional derivative $\rho(x, t) = 0.8 + 0.2t/(TL)$. 

<table>
<thead>
<tr>
<th>Time</th>
<th>Explicit Scheme</th>
<th>Implicit Scheme</th>
<th>Crank-Nicholson Scheme</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t = 0.1$</td>
<td>0.012E-2</td>
<td>0.574E-3</td>
<td>0.418E-3</td>
</tr>
<tr>
<td>$t = 0.2$</td>
<td>0.356E-2</td>
<td>1.304E-3</td>
<td>0.698E-3</td>
</tr>
<tr>
<td>$t = 0.3$</td>
<td>1.257E-2</td>
<td>2.43E-3</td>
<td>0.568E-3</td>
</tr>
<tr>
<td>$t = 0.4$</td>
<td>2.69E-2</td>
<td>4.28E-3</td>
<td>0.233E-3</td>
</tr>
<tr>
<td>$t = 0.5$</td>
<td>5.42E-2</td>
<td>7.11E-3</td>
<td>0.207E-3</td>
</tr>
</tbody>
</table>

Table 3. Absolute errors of the explicit scheme, the implicit scheme and the Crank-Nicholson scheme for VO fractional diffusion equation at $x = 5.0$. The time step length $\tau = 0.01$, the space step size $h = 0.1$ and the VO function of time fractional derivative $\rho(x, t) = 0.8 + 0.2t/(TL)$, where $L = 10.0$, $T = 1.0$. 

<table>
<thead>
<tr>
<th>Time</th>
<th>Explicit Scheme</th>
<th>Implicit Scheme</th>
<th>Crank-Nicholson Scheme</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t = 0.1$</td>
<td>0.009E-2</td>
<td>0.558E-3</td>
<td>0.442E-3</td>
</tr>
<tr>
<td>$t = 0.2$</td>
<td>0.353E-2</td>
<td>1.186E-3</td>
<td>0.813E-3</td>
</tr>
<tr>
<td>$t = 0.3$</td>
<td>1.181E-2</td>
<td>2.132E-3</td>
<td>0.885E-3</td>
</tr>
<tr>
<td>$t = 0.4$</td>
<td>2.78E-2</td>
<td>3.596E-3</td>
<td>0.498E-3</td>
</tr>
<tr>
<td>$t = 0.5$</td>
<td>5.085E-2</td>
<td>5.680E-3</td>
<td>0.578E-3</td>
</tr>
</tbody>
</table>

Fig. 3. The exact solution curve and numerical results by the explicit scheme, the explicit scheme and the Crank-Nicholson scheme at $t = 0.05$, $K = 0.01$. The time step length $\tau = 0.1$, the space step size $h = 0.1$. The VO function of time fractional derivative $\rho(x, t) = 0.8 + 0.2t/(TL)$, where $L = 10.0$, $T = 1.0$. 

Table 3. Absolute errors of the explicit scheme, the implicit scheme and the Crank-Nicholson scheme for VO fractional diffusion equation at $x = 5.0$. The time step length $\tau = 0.01$, the space step size $h = 0.1$ and the VO function of time fractional derivative $\rho(x, t) = 0.8 + 0.2t/(TL)$, where $L = 10.0$, $T = 1.0$. 

obtain that the accuracy of the finite difference schemes has close relationship with the evolution process of the VO. Generally speaking, if the VO function changes slightly with time and space, the accuracy will be better. On the other hand, if the VO changes sharply with time or space, the absolute errors will become larger. In addition, the absolute error will decrease by refining grids.

Example 2. Here we consider VO time fractional diffusion equation without source term, which can

Fig. 4. The evolution curve of absolute error at $x = 5.0$, $K = 0.01$. The time step length $\tau = 0.01$, the space step size $h = 0.1$. The VO function of time fractional derivative $\rho(x, t) = 0.8 + 0.2t/(TL)$, where $L = 10.0$, $T = 1.0$. 

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read as follows
\[
\begin{align*}
&c_1 \partial_t D_t^{\alpha}(x,t) u(x,t) = K \partial_x^2 u(x,t), \quad x \in (0,L), \quad t \in (0,T], \\
&u(x,0) = \sin \left(\frac{x \pi}{L}\right), \quad x \in [0,L], \\
&w(0,t) = 0, \quad u(L,t) = 0, \quad t \in [0,T], \\
\end{align*}
\] (60)

where \(0 < \alpha(x,t) \leq 1\), for \(\forall (x,t)\).

The accuracy of the implicit scheme is rather worse compared with the implicit scheme and the Crank-Nicholson scheme, which has been verified by Example 1. So, in this example, we only employ the implicit scheme and the Crank-Nicholson scheme to perform numerical simulation. From Fig. 6 and Table 4, it can be observed that the numerical result by the implicit scheme is consistent well with that by the Crank-Nicholson scheme. Both of them have captured correct evolution trend of model (60) that the diffusion pattern shifts from diffusion model with \(\alpha(x,t) = 0.8\) to the one with \(\alpha(x,t) = 1.0\). By comparing Figs. 6 and 7, we observe that the VO model (60) with \(\alpha(x,t) = 0.8 + 0.2t/T\) gives faster decay than that in model with \(\alpha(x,t) = 0.8 + 0.2t/(LT)\), which is in good agreement with the analytical and numerical solutions provided in [Sun et al., 2009].

Table 4. The numerical result of VO time fractional diffusion equation (60) by the Implicit Scheme (ImS) and the Crank-Nicholson Scheme (CNS), in which \(K = 0.01\) and \(\alpha(x,t) = 0.8 + 0.2t/(LT)\). The time step is \(\tau = 0.01\), the space step size \(h = 0.01\), where \(L = 1.0, T = 10.0\).

<table>
<thead>
<tr>
<th>Scheme (Time)</th>
<th>(x = 0.2)</th>
<th>(x = 0.4)</th>
<th>(x = 0.6)</th>
<th>(x = 0.8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ImS ((t = 4.0))</td>
<td>0.391525</td>
<td>0.391525</td>
<td>0.391525</td>
<td>0.391525</td>
</tr>
<tr>
<td>CNS ((t = 4.0))</td>
<td>0.391525</td>
<td>0.391525</td>
<td>0.391525</td>
<td>0.391525</td>
</tr>
<tr>
<td>CNS ((t = 8.0))</td>
<td>0.263964</td>
<td>0.263964</td>
<td>0.263964</td>
<td>0.263964</td>
</tr>
</tbody>
</table>

Fig. 6. The evolution curve of \(u(x=0.5,t)\) by the implicit scheme and the Crank-Nicholson scheme at \(x = 0.5\), in which \(K = 0.01\) and \(\alpha(x,t) = 0.8 + 0.2t/(LT)\). The time step length \(\tau = 0.01\), the space step size \(h = 0.01\), where \(L = 1.0, T = 10.0\).

Fig. 7. The evolution curve of \(u(x=0.5,t)\) by the implicit scheme and the Crank-Nicholson scheme at \(x = 0.5\), in which \(K = 0.01\) and \(\alpha(x,t) = 0.8 + 0.2t/(LT)\). The time step length \(\tau = 0.01\), the space step size \(h = 0.01\), where \(L = 1.0, T = 10.0\).
Finite Difference Schemes for Variable-Order Time Fractional Diffusion Equation

6. Conclusion
In this paper, three finite difference schemes, i.e. the explicit scheme, the implicit scheme and the Crank–Nicholson scheme have been employed for solving the VO time fractional diffusion equation in a bounded domain. We have presented a unifying approach to finite difference schemes for time fractional diffusion equations of both the constant-order and VO cases. It has been proved that the explicit scheme is conditionally stable and convergent, the implicit scheme and the Crank–Nicholson scheme are unconditionally stable and convergent. The numerical examples have validated our analysis about stability, convergence and have also shown some properties of VO time fractional diffusion equation model.

It should be pointed that, though we have only considered VO time fractional diffusion equation in this article, the suggested methods can be possibly extended to solve VO time fractional advection-diffusion equation and random-order time fractional diffusion equation [Sun et al., 2011].

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References


